Corner transfer matrices in statistical mechanics

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Abstract
Corner transfer matrices are a useful tool in the statistical mechanics of simple
two-dimensional models. They can be a very effective way of obtaining series
expansions of unsolved models, and of calculating the order parameters of
solved ones. Here we review these features and discuss the reason why the
method fails to give the order parameter of the chiral Potts model.

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1. Introduction

In an early work of the author [1], he evaluated numerically a sequence of approximations
for the free energy (or entropy) of the monomer–dimer system on the square lattice, using
a variational approximation for the eigenvalues and eigenvectors of the row-to-row transfer
matrix. His interest was to see if the model exhibited a phase transition as the density of
dimers increased from zero to the close packed value of one-half. No such transition was
observed: the model is critical only at close packing, where it can be solved exactly [20]. This
was reflected in the poorer convergence of the approximations as close packing is approached.

The result may have been negative, but the work did lead naturally to the development of
the useful concept of 'corner transfer matrices'.

These matrices build up the lattice by rotations about the centre. For an infinite lattice
they are of infinite dimension, but for the zero-field Ising model, one can write them in terms
of the spinor operators (or Clifford algebra) introduced by Kaufman [21] and diagonalize them
exactly [2, 25]. The results were surprisingly simple, and provided a reasonably direct way of
obtaining the spontaneous magnetization (the order parameter) of the Ising model, compared
with the previous verifications of the Onsager–Kaufman result [23] by Yang [26], Montroll,
Potts and Ward [22] and others.

It was also realized that the corner transfer matrices satisfy certain equations. Again, to
give exact results the matrices and equations must be infinite dimensional, but a finite truncation
corresponds to a variational approximation of the type used for the monomer–dimer system.
Such truncations can give useful approximations to the exact results, even for models which

have not been solved exactly. For instance, Baxter and Enting [12] were able to obtain 24 terms in the low-temperature series expansion of the Ising model in a field, using only 15 by 15 matrices. This was a great improvement on the 12-term series expansions previously obtained [24].

Here we present these equations for a quite general 'edge-interaction' model, where spins live on sites of the square lattice and the interactions are between adjacent spins. We discuss the simplifications that arise for a 'solvable' model, i.e. one whose Boltzmann weights satisfy the star-triangle relation. We emphasize the importance of the 'rapidity-difference' property and indicate how it ensures that the corner transfer matrices commute and have a very simple eigenvalue spectrum. This makes it easy to obtain the order parameters of such models.

Finally we discuss the chiral Potts model, and indicate how the lack of a rapidity difference property for that model prevents its solution by the same corner transfer matrix techniques.

For definiteness, here we consider only Z_N-invariant models (such as the Ising, self-dual Potts and chiral Potts models), where each spin can take N values and the Boltzmann weights depend only on the difference (modulo N) of the spin values. The extension to more general edge-interaction models (perhaps including site weight functions as in the solvable Kashiwara–Miwa model [15, 17, 19]) is straightforward.

2. Square lattice edge-interaction models

In figure 1 we have drawn the square lattice $\mathcal{L}$ diagonally, denoting sites by circles and edges by solid lines. On each site i there is a 'spin' σ_i, taking some discrete set of N values, say $0, 1, \ldots, N-1$. Spins on adjacent sites i and j, with j above and to the right of i, interact with the Boltzmann weight function $W(\sigma_i - \sigma_j)$, as indicated. Similarly, spins on sites k and l, with l above and to the left of k, interact with the weight function $\overline{W}(\sigma_k - \sigma_l)$. The partition function is

$$Z = \sum_\sigma \prod_{i,j} W(\sigma_i - \sigma_j) \prod_{k,l} \overline{W}(\sigma_k - \sigma_l), \qquad (2.1)$$

the sum being over all values of all the spins on the lattice, the first product over all SW–NE edges, the second over all SE–NW edges.

If the lattice has L sites, we expect the limit

$$\kappa = \lim_{L \to \infty} Z^{1/L} \qquad (2.2)$$

to exist, and to be independent of the manner in which $\mathcal{L}$ becomes large, so long as it becomes infinite in all directions.

In figure 1 we have singled out a central site with spin a. The average of any function $f(a)$ is defined as

$$\langle f(a) \rangle = Z^{-1} \sum_\sigma f(a) \prod_{i,j} W(\sigma_i - \sigma_j) \prod_{k,l} \overline{W}(\sigma_k - \sigma_l). \qquad (2.3)$$

We also expect any such average to tend to a limit, provided we impose suitable conditions on the values of the boundary spins, and take a limit where a becomes infinitely far from any boundary.

2.1. Corner transfer matrices

Here we define the corner transfer matrices (and the associated row and column matrices) for a general anisotropic edge-interaction model.
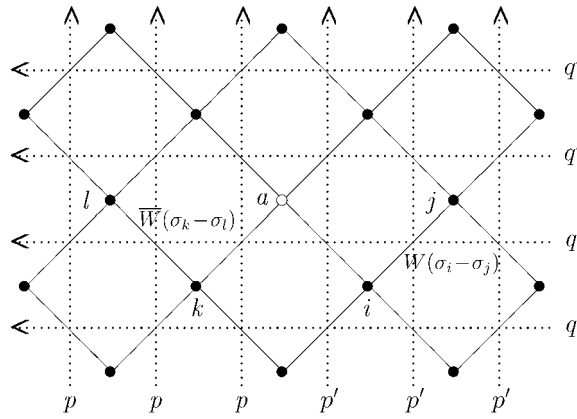
Figure 1. The square lattice (solid lines, drawn diagonally), showing a central spin a and the (dotted) rapidity lines.
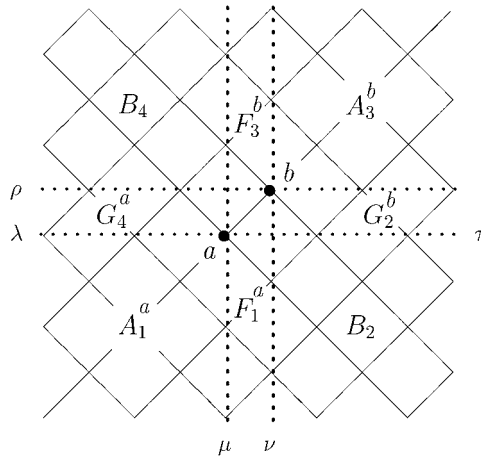
Figure 2. The parts of the lattice corresponding to the various matrices A, B, F, G.

We have again drawn the diagonal square lattice in figure 2 and have singled out a central edge linking two sites with spins a and b. We have then used dotted lines to divide the lattice into nine parts: the central edge, four 'corners' labelled respectively A_1^a, B_2, A_3^b, B_4, and four columns or rows F_1^a, G_2^b, F_3^b, G_4^a.

Take all the spins on the outermost boundary of the lattice to have some fixed value[1], say zero. Let λ be the set of spins on the dotted line to the left of the spin a and μ the set of spins on the dotted line below a. Consider all the edges of the lattice below λ and to the left of μ, i.e. all the edges in the lower-left corner of the lattice. Take the product over all such edges of the appropriate weight functions W or $\overline{W}$, and sum over all the spins internal to that corner. The resulting partition function of the corner will be a function of a, λ and μ. Write it as $\left(A_1^a\right)_{\lambda,\mu}$.

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[1] For an ordered system, the boundary spins should be set to values consistent with the ground state of the particular phase under consideration.
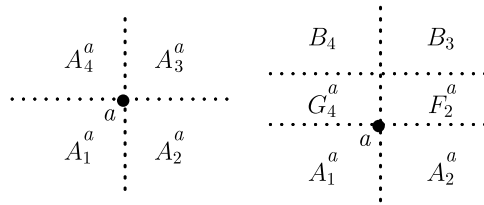
Figure 3. The partition functions Z_A, Z_2. We have omitted the edges of $\mathcal{L}$, but their positions can be deduced from the full circle denoting the lattice spin a.

Similarly, let $\left(F_1^a\right)_{\mu,\nu}$ be the product of the weight functions of the edges between μ and ν. Let $(B_2)_{\nu,\tau}$ be the partition function of the lower-right corner of the lattice, i.e. the portion to the right of μ and below τ and so on. Then the partition function of the complete lattice is $Z = Z_W$, where

$$Z_W = \sum W(a-b)\left(A_1^a\right)_{\lambda,\mu}\left(F_1^a\right)_{\mu,\nu}(B_2)_{\nu,\tau}\ldots\left(G_4^a\right)_{\rho,\lambda}, \tag{2.4}$$

the sum being over all values of a, b and all the spins on the dotted lines of figure 2.

Plainly we can regard $\left(A_1^a\right)_{\lambda,\mu}$ as the element (λ, μ) of a matrix A_1^a, $\left(F_1^a\right)_{\mu,\nu}$ as the element (μ, ν) of a matrix F_1^a, etc. Then (2.4) simplifies to

$$Z_W = \sum_{a,b} W(a-b)\ \text{Trace}\ A_1^a F_1^a B_2 G_2^b A_3^b F_3^b B_4 G_4^a, \tag{2.5}$$

the sum now being only over the spins a, b.

There are other ways of building up Z using the matrices A_i, B_i, F_i, G_i as building blocks, most of them being simpler than (2.5). They are

$$Z_{\overline{W}} = \sum_{a,b} \overline{W}(a-b)\ \text{Trace}\ B_1 G_1^a A_2^a F_2^a B_3 G_3^b A_4^b F_4^b,$$

$$Z_A = \sum_a \text{Trace}\ A_1^a A_2^a A_3^a A_4^a, \qquad Z_B = \text{Trace}\ B_1 B_2 B_3 B_4,$$

$$Z_i = \sum_a \text{Trace}\ A_i^a F_i^a B_{i+1} B_{i+2} G_{i+2}^a A_{i+3}^a,$$

where $i = 1, \ldots, 4$ and the suffixes containing i are to be interpreted modulo 4. We have indicated how Z_A and Z_2 are constructed in figure 3.

Note that F_i^a always follows A_i^a and precedes B_{i+1}, while G_i^a follows B_i and precedes A_{i+1}^a.

If we keep the sizes of the blocks corresponding to the A_i, B_i, F_i, G_i matrices fixed (and mutually consistent), then Z_A, Z_B correspond to the smallest lattices, Z_W, $Z_{\overline{W}}$ to the largest, and $Z_1, \ldots Z_4$ to intermediate ones. Using (2.2), we find that in the limit of the lattices becoming large, the partition function per site is

$$\kappa = \frac{Z_A Z_B Z_W Z_{\overline{W}}}{Z_1 Z_2 Z_3 Z_4}. \tag{2.6}$$

We shall find it convenient to define four Boltzmann weight functions $U_1, \ldots, U_4$ by

$$U_1(n) = W(n), \qquad U_2(n) = \overline{W}(n),$$
$$U_3(n) = W(-n), \qquad U_4(n) = \overline{W}(-n) \tag{2.7}$$

and to set

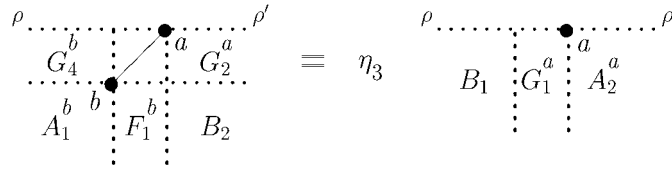$$\tilde{Z}_1 = \tilde{Z}_3 = Z_W, \qquad \tilde{Z}_2 = \tilde{Z}_4 = Z_{\overline{W}}. \tag{2.8}$$

Figure 4. Graphical representation of the elements (ρ, ρ') of the two sides of equation (2.9a) for $i = 3$.

Equations [3, 4] for the corner transfer matrices can be obtained by requiring that the expression (2.6) be stationary with respect to variations in the matrices A_i, B_i, F_i, G_i. This variational principle leads to the four equations

$$\sum_b U_i(a-b)G_{i+1}^b A_{i+2}^b F_{i+2}^b B_{i+3} G_{i+3}^a = \eta_i B_{i+2} G_{i+2}^a A_{i+3}^a, \tag{2.9a}$$

$$\sum_b U_{i-1}(a-b)F_{i-1}^a B_i G_i^b A_{i+1}^b F_{i+1}^b = \eta_i' A_i^a F_i^a B_{i+1}, \tag{2.9b}$$

$$F_i^a B_{i+1} B_{i+2} G_{i+2}^a = \xi_i A_{i+1}^a A_{i+2}^a, \tag{2.9c}$$

$$\sum_b G_{i-2}^b A_{i-1}^b A_i^b F_i^b = \xi_i' B_{i-1} B_i. \tag{2.9d}$$

Our notation is now such that rotating the lattice through $90°$ is equivalent to incrementing i by 1 in all suffixes.

Pre-multiplying both sides of (2.9a) by $A_i^a F_i^a B_{i+1}$ and summing over a, we find that

$$\eta_i = \tilde{Z}_i / Z_i. \tag{2.10}$$

Similarly,

$$\eta_i' = \tilde{Z}_{i+1}/Z_i, \qquad \xi_i = Z_i/Z_A, \qquad \xi_i' = Z_i/Z_B \tag{2.11}$$

and, for instance,

$$\kappa = \eta_i \eta_{i+2}'/(\xi_{i+1}\xi_{i+3}'). \tag{2.12}$$

Equations (2.9a)–(2.9d) can be represented graphically, each side being a semi-lattice, e.g. for $i = 3$ equation (2.9a) can be represented as in figure 4.

We can only expect these equations to be exact (for an infinite system) when the matrices are infinitely dimensional. However, if we take them to finite-dimensional, then the fact that (2.9a)–(2.9d) are derived from the variational principle for κ ensures that these equations are mutually consistent and that they will define the A_i, B_i, F_i, G_i (to within irrelevant similarity transformations). There will be many solutions—a test of the utility of any particular solution is that it should reproduce as many of the largest eigenvalues of $A_1^a A_2^a A_3^a A_4^a$ and $B_1 B_2 B_3 B_4$ as possible for that truncation.

Such truncations can provide a powerful tool for performing numerical or series expansion calculations on otherwise unsolved models [9]. Indeed, it was such a calculation that led to the exact solution of the hard hexagon model [5, 14].

3. Solvable models

We now focus our attention on models that are known to be solvable and satisfy the star-triangle relation. Their corner transfer matrix have special simplifying properties. These

properties are true only in the infinite-dimensional limit, but for any given truncation they will be approximately true, in particular they will hold to appropriate orders in a low-temperature series expansion.

In general, if a model satisfies the star-triangle relation, then its weights $W(n)$, $\overline{W}(n)$ depend on two variables p and q, called 'rapidities'. The variable p is associated with the vertical direction of the lattice $\mathcal{L}$, q with the horizontal direction. We display this dependence by writing the weights as $W_{pq}(n)$, $\overline{W}_{pq}(n)$. The star-triangle relation relates three sets of Boltzmann weights, with rapidities (p, q), (p, r), (q, r). It is

$$\sum_d \overline{W}_{qr}(b-d) W_{pr}(a-d) \overline{W}_{pq}(d-c) = R_{pqr} W_{pq}(a-b) \overline{W}(b-c) W_{qr}(a-c), \qquad (3.1)$$

for all p, q, r and values of the three external spins a, b, c. The sum is over all values of the internal spin d.

A second relation must also hold, where all the spin-difference arguments in (3.1) are negated, so that $b - d$ becomes $d - b$, $a - b$ becomes $b - a$, etc. Provided we impose cyclic boundary conditions, (3.1) ensures that the row-to-row transfer matrices of two models, one with rapidities (p, q), the other with (p, r), commute [4]. Thus it is natural to allow q, r to vary from row to row of the lattice $\mathcal{L}$, and in figure 1 we have used this freedom to associate two rapidities q, q' with the horizontal rows of the lattice. Moreover, this is true even if the rapidities on the dotted vertical lines of figure 1 are different, e.g., if the lines to the left of a have rapidity p, while those to the right have rapidity p', as shown.

More generally, we can allow a different rapidities p for each of the dotted vertical lines in figure 1, and a different rapidity q for each horizontal line. The Boltzmann weight of any edge of $\mathcal{L}$ is $W_{pq}(n)$ or $\overline{W}_{pq}(n)$, where the p and q are the rapidities of the dotted lines intersecting that edge.

It follows that the eigenvectors of these transfer matrices are *independent* of horizontal rapidities such as the q, q' shown in figure 1. The correlations of spins lying within a single row of $\mathcal{L}$ will therefore also be independent of q, q'.

But (to within an overall irrelevant normalization factor) the expressions represented by figure 4 are precisely such correlations, involving only the spin sets ρ, ρ' and the spin a on the upper edge of each side. This implies that

$$\left(B_1 G_1^a A_2^a \right)_{\rho, \rho'} = \text{independent of } q, q'. \qquad (3.2)$$

There is a problem with this argument: we are using fixed-spin boundary conditions, not cyclic ones. However, provided the conditions are commensurate with the ground state of the system, we expect (3.2) to be true in the limit when the lattice is large and only a finite number of the spins in ρ, ρ' differ from the ground-state values. This is how we construct the infinite-dimensional matrices, so we believe (3.2) to be true in this limit.

For i odd, it follows that the matrices

$$A_i^a A_{i+1}^a, \qquad B_i B_{i+1}, \qquad A_i^a F_i^a B_{i+1}, \qquad B_i G_i^a A_{i+1}^a \qquad (3.3)$$

are independent of horizontal rapidities such as q, q'. Similarly, for i even, they are independent of vertical rapidities such as p, p'.

Each matrix will depend only on the rapidities p, q of that particular, corner, row or column. For instance, B_1 will depend only on the rapidities of the lower-left corner. For each matrix we take all the p's to be that same, and all the q's, and write the matrix as a function of p, q, e.g.

$$B_1 = B_1(p, q).$$

For i odd, each of the expressions (3.3) is a product of matrices corresponding to blocks of $\mathcal{L}$ that occupy the same rows of $\mathcal{L}$. The horizontal rapidity q must therefore be the same

for each matrix within the expression, but the vertical p rapidities may be different. Thus, for instance, the above argument gives

$$B_1(p, q)B_2(p', q) = \text{independent of } q$$

for all p, p', q. Fixing p' and assuming the B_i matrices to be invertible, it follows that there exist invertible matrices $R(p)$, $S(q)$ such that

$$B_1(p, q) = R(p)S(q).$$

This is a factorization property. All the corner matrices A_i^a, B_i similarly factor and we find that

$$
\begin{aligned}
A_i^a(p, q) &= \left[X_{i-1}^a(\tilde{p}_i)\right]^{-1} \mathcal{A}_i^a X_i^a(\tilde{q}_i), \\
B_i(p, q) &= [Y_{i-1}(\tilde{p}_i)]^{-1} \mathcal{B}_i Y_i(\tilde{q}_i), \\
F_i^a(p, q) &= \left[X_i^a(\tilde{q}_i)\right]^{-1} \mathcal{F}_i^a(\tilde{p}_i) Y_i(\tilde{q}_i), \\
G_i^a(p, q) &= [Y_i(\tilde{q}_i)]^{-1} \mathcal{G}_i^a(\tilde{p}_i) X_i^a(\tilde{q}_i),
\end{aligned}
\tag{3.4}
$$

where $\tilde{p}_i = p$, $\tilde{q}_i = q$ if i is odd; and $\tilde{p}_{i+1} = \tilde{q}_i$, $\tilde{q}_{i+1} = \tilde{p}_i$ for all i. Note that the matrices $\mathcal{A}_i^a$, $\mathcal{B}_i$ are independent of p and q, while $\mathcal{F}_i^a$, $\mathcal{G}_i^a$ depend only on the single rapidity $\tilde{p}_i$.

The choice of the X_i, Y_i is not unique: each can be pre-multiplied by different constant invertible matrices C_i, D_i and the $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{F}_i$, $\mathcal{G}_i$ then adjusted accordingly so as to ensure that (3.4) remains satisfied. However, this does *not* change the eigenvalues of

$$\mathcal{M}^a = \mathcal{A}_1^a \mathcal{A}_2^a \mathcal{A}_3^a \mathcal{A}_4^a, \qquad \mathcal{N} = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4.$$

These matrices define Z_A and Z_B and play an important role in the calculations: for instance, the order parameters are

$$\langle \omega^{ja} \rangle = \sum_a \omega^{ja} \, \text{Trace} \, \mathcal{M}^a \left/ \sum_a \text{Trace} \, \mathcal{M}^a \right. \tag{3.5}$$

where $\omega = e^{2i\pi/N}$ and $j = 1, \dots, N - 1$.

Substituting these forms into equations (2.9a)–(2.9d), all the X_i, Y_i matrices cancel out, so the effect is to replace the matrices A_i, B_i, F_i, G_i by $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{F}_i$, $\mathcal{G}_i$, where $\mathcal{A}_i$, $\mathcal{B}_i$ are rapidity independent and $\mathcal{F}_i$, $\mathcal{G}_i$ depend only on the single rapidity $\tilde{p}_i$.

3.1. Properties and symmetries

There are various properties and symmetries that relate the corner transfer matrices. For the Ising, self-dual Potts and chiral Potts models [13], it is true that

$$W_{pp}(n) = 1, \qquad \overline{W}_{pp}(n) = \delta_{n,0} \tag{3.6}$$

and there exist rotation and reflection transformations $p \to Rp$, $p \to Sp$, such that

$$\overline{W}_{pq}(n) = W_{q,Rp}(n), \qquad W_{pq}(n) = \overline{W}_{q,Rp}(-n), \tag{3.7}$$

$$W_{pq}(n) = W_{Sq,Sp}(n), \qquad \overline{W}_{pq}(n) = \overline{W}_{Sq,Sp}(-n). \tag{3.8}$$

If we write $U_i(n)$ more explicitly as $U_i(p, q|n)$, it follows that $U_i(q, Rp|n) = U_{i+1}(p, q|n)$ and

$$A_i^a(p, p) = B_i(p, p) = \mathbf{1} \qquad \text{for} \quad i = 1 \text{ and } 3, \tag{3.9}$$

$$A_i^a(q, Rp) = A_{i+1}^a(p, q), \dots, G_i^a(q, Rp) = G_{i+1}^a(p, q), \tag{3.10}$$

and, for instance,

$$B_i(Sq, Sp) = B_{6-i}(p, q)^T. \tag{3.11}$$

Replacing p, q by q, Rp is simply equivalent to rotating the lattice through $90°$, i.e. incrementing the suffixes i by one.

These equations impose many restrictions of the matrices $X_i, Y_i, \mathcal{A}_i, \mathcal{B}_i, \mathcal{F}_i, \mathcal{G}_i$. We shall not explore all of them here, but note that they do imply that there exists a matrix function $X^a(p)$ such that we can choose

$$X_1^a(p) = X_4^a(p) = X^a(p), \qquad X_2^a(p) = X_3^a(p) = \left(\mathcal{A}_2^a\right)^{-1} X^a(p),$$
$$\mathcal{A}_1^a = \mathcal{A}_3^a = \mathbf{1} \tag{3.12}$$

$$A_1^a(p, q) = [X^a(p)]^{-1} X^a(q), \qquad A_2^a(p, q) = [X^a(q)]^{-1} X^a(Rp),$$
$$A_3^a(p, q) = [X^a(Rp)]^{-1} X^a(Rq), \qquad A_4^a(p, q) = [X^a(Rq)]^{-1} X^a(R^2 p), \tag{3.13}$$

where

$$X^a(R^2 p) = \mathcal{M}^a X^a(p). \tag{3.14}$$

Corresponding equations hold for the matrices $Y_i, \mathcal{B}_i, B_i, \mathcal{N}_i$, with no superfixes a.

4. Ising model

We have seen how the star-triangle relation implies that row-to-row transfer matrices commute and corner transfer matrices factor. To proceed further we need an important property, namely that for many models we can choose the rapidity variables p, q so that

$$W_{pq}(n), \overline{W}_{pq}(n) = \text{functions only of } q - p \text{ and } n. \tag{4.1}$$

We call this the *rapidity difference* property. It is intimately connected with the fact that for such models the Boltzmann weights live on algebraic curves of genus 0 or 1, so can be naturally be parametrized in terms of Jacobi elliptic functions. One had become so used to this property that it was taken for granted. It was the discovery of the chiral Potts model [13], which does *not* have this property, that made its importance obvious.

The Ising model is a two-state model (i.e. $N = 2$) of the type discussed above. Taking $W_{pq}(0) = \overline{W}_{pq}(0) = 1$, its other Boltzmann weights are [8]

$$W_{pq}(1) = \mathrm{e}^{-2J} = k' \operatorname{scd}(K - u), \tag{4.2}$$

$$\overline{W}_{pq}(1) = \mathrm{e}^{-2\overline{J}} = k' \operatorname{scd}(u), \tag{4.3}$$

where

$$u = u_q - u_p, \tag{4.4}$$

and the function $\operatorname{scd}(u)$ is defined by

$$\operatorname{scd}(u) = \operatorname{sn}(u/2)/[\operatorname{cn}(u/2)\operatorname{dn}(u/2)],$$

$\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$ being the Jacobi elliptic functions of argument u and modulus k.

Then, setting $k' = (1 - k^2)^{1/2}$, we have

$$\sinh 2J = \frac{\operatorname{sn} u}{\operatorname{cn} u}, \qquad \sinh 2\overline{J} = \frac{\operatorname{cn} u}{k' \operatorname{sn} u} \tag{4.5}$$

and $\sinh 2J \sinh 2\overline{J} = 1/k'$. The conjugate modulus k' is small at low temperatures, increasing to one at criticality. Within this range the model has ferromagnetic order. The Boltzmann weights satisfy (3.7) if we define

$$u_{Rp} = u_p + K, \tag{4.6}$$

they are real and positive provided $u = u_q - u_p$ is real and

$$0 < u_q - u_p < K, \tag{4.7}$$

K, K' being the usual complete elliptic integrals. Note that this restriction remains satisfied if we replace p, q by Rq, p, when $u_q - u_p$ becomes $K - u_q + u_p$.

The relation (4.4) manifests the difference property. It has far-ranging consequences, in fact it enables us to calculate the spontaneous magnetization, i.e. the order parameter (3.5) with $j = 1$ [4, chapter 13].

To see this, note that we write the first equation (3.13) as

$$A_1^a(u_q - u_p) = [X(u_p)^a]^{-1} X(u_q)^a. \tag{4.8}$$

Replacing p, q by q, r and multiplying, this gives

$$A_1^a(u_q - u_p) A_1^a(u_r - u_q) = A_1^a(u_r - u_p)$$

i.e.

$$A_1^a(u) A_1^a(v) = A_1^a(u + v), \tag{4.9}$$

for arbitrary u, v. Interchanging u, v, it follows that

$$A_1^a(u) A_1^a(v) = A_1^a(v) A_1^a(u), \tag{4.10}$$

so $A_1^a(u)$, $A_1^a(v)$ *commute*, for all u, v.

There is therefore a similarity transformation, independent of u, that diagonalizes $A_1^a(u)$. Let us assume this transformation has been applied and take $A_1^a(u)$ to be diagonal. Then it follows from (4.9) that each diagonal element of $A_1^a(u)$ is an *exponential* function of u, i.e.

$$\left[A_1^a(u)\right]_{i,i} = \exp(-n_i u), \tag{4.11}$$

where n_i is some number.

Allow $u = u_q - u_p$ to be an arbitrary complex number. Since

$$\mathrm{scd}(u + 2\mathrm{i}K') = -\mathrm{scd}(u),$$

it follows from (4.2) that incrementing u by $2\mathrm{i}K'$ merely negates $W_{pq}(1)$ and $\overline{W}_{pq}(1)$. From series expansions, we expect $A_1^a(u)$, and indeed all the functions we have defined in the thermodynamic limit of a large lattice, to be an analytic function of u within the extended physical domain $0 < \Re(u) < K$, which implies

$$A_1^a(u + 2\mathrm{i}K') = (-1)^a A_1^a(u). \tag{4.12}$$

It follows that

$$n_i = \pi m_i/(2K'), \tag{4.13}$$

where m_i is an even integer for $a = 0$, an odd integer for $a = 1$.

If we take the low-temperature limit $k' \to 0$, keeping u in the vicinity of $K/2$, then to leading order in $q' = \mathrm{e}^{-\pi K/K'}$,

$$W_{pq}(1) = \mathrm{e}^{-\pi u/(2K')}, \qquad \overline{W}_{pq}(1) = \mathrm{e}^{\pi(u-K)/(2K')}. \tag{4.14}$$

We find that $A_1^a(u)$, as originally defined before (2.4), is already diagonal to leading order. Let the spins to the left of and including a in figure 2 be $\ldots, \lambda_2, \lambda_1, a$, where λ_1 is next to

a, λ_2 is next to λ_1, etc. Then the corresponding diagonal element of $A_1^a(u)$ is indeed of the form (4.11), (4.13), the index i being replaced by the spin set λ, i.e.

$$A_1^a(u)_{\lambda, \lambda} = \exp\{-\pi u m_\lambda / (2K')\}, \tag{4.15}$$

where

$$m_\lambda = |a - \lambda_1| + 3|\lambda_1 - \lambda_2| + 5|\lambda_2 - \lambda_3| + \cdots. \tag{4.16}$$

We have obtained this result in the low-temperature limit $k' \to 1$. However, for all k' the m_λ must be integers and we do not expect them to change discontinuously as k' increases to unity. Hence we expect the formulae (4.15), (4.16) for the diagonal elements of $A_1^a(u)$ to be *exact* for $0 < k' < 1$.

Set $u_1 = u_2 = u_q - u_p$ and $u_2 = u_4 = K - u_q + u_p$. Then from (3.10),

$$A_i^a(p, q) = A_1^a(u_i), \tag{4.17}$$

so these matrices, together with the product

$$M^a = A_1^a(p, q) A_2^a(p, q) A_3^a(p, q) A_4^a(p, q) \tag{4.18}$$

are diagonalized by the same transformation that diagonalizes $A_1^a(u)$, and

$$M_{\lambda, \lambda}^a = x^{m_\lambda}, \tag{4.19}$$

where $x = q' = e^{-\pi K / K'}$.

These are also the elements of the diagonalized matrix $\mathcal{M}^a$: we see that our calculation is consistent with the independence of $\mathcal{M}^a$ on the rapidity variables p, q.

It is now straightforward to calculate the spontaneous magnetization from (3.5). First set $\lambda_0 = a$ and

$$l_i = |\lambda_i - \lambda_{i+1}|, \qquad i \geqslant 0, \tag{4.20}$$

so $l_i = 0$ if $\lambda_i = \lambda_{i+1}$, otherwise $l_i = 1$. Take $\lambda_i = 0$ for sufficiently large i, then to modulo 2,

$$a = \lambda_0 + \lambda_1 + \lambda_2 + \cdots,$$

and

$$m_\lambda = \lambda_0 + 3\lambda_1 + 5\lambda_2 + \cdots.$$

From (3.5) and (4.19),

$$\langle (-1)^a \rangle = \sum_l (-1)^a x^{m_\lambda} \Big/ \sum_l x^{m_\lambda}. \tag{4.21}$$

Here a, m_λ are the linear expressions immediately above and the sums are over $l_0, l_1, l_2, \ldots = 0, 1$, without restriction. Each sum therefore factors into a product of individual sums over $l_0, l_1, l_2, \ldots$, giving

$$\langle (-1)^a \rangle = \frac{(1 - x)(1 - x^3)(1 - x^5) \cdots}{(1 + x)(1 + x^3)(1 + x^5) \cdots}. \tag{4.22}$$

Using known elliptic function formulae [16, 8.197.4], and [4, 15.1.4b] the RHS is $k^{1/4}$, so

$$\langle (-1)^a \rangle = k^{1/4} = (1 - k'^2)^{1/8} \tag{4.23}$$

which is the famous result announced by Onsager in 1949 [23] and derived by Yang in 1952 [26].

5. Summary

We have explicitly shown how the star-triangle relation implies that the corner transfer matrices factor, causing the reduced matrix functions $\mathcal{A}_i^a$, $\mathcal{B}_i$, $\mathcal{F}_i^a$, $\mathcal{G}_i^a$ to depend on at most one of the two rapidity variables. The rotation (and reflection) relations enable one to express the matrices, for $i = 1, \ldots, 4$, in terms of the single $i = 1$ case.

Curiously, this by itself does not seem enough to enable one to calculate quantities such as the order parameters: if one also has the rapidity difference property (4.1), then the corner transfer matrices commute and can be simultaneously reduced to diagonal form. The eigenvalues are necessarily exponentials in $u = u_q - u_p$ and are periodic of period $2iK'$: this means they are defined by certain integers m_λ, and these integers can be obtained from a low-temperature limit. The order parameters follow immediately.

It still seems strange that the method should fail for a model such as the chiral Potts model, which does satisfy the star-triangle relation, so in that sense is 'solvable'. As yet the method has only yielded finite series expansions [6, 7].

The order parameters have now been obtained by a specialization of the Jimbo, Miwa, Nakayashiki method [10, 11, 18], but it would be interesting to rescue the corner transfer matrix technique for this model. For the models with the difference property, diagonalizing the matrices leads directly to the Jacobi elliptic parametrization. For the chiral Potts model, is there any way of expressing the matrices that leads to an analogous parametrization?

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